Univerzitet u Zenici
Filozofski fakultet
Odsjek: Matematika i informatika
Zenica, 09.07.2015.

Linearna algebra, pismeni ispit

1. Neka je $\mathcal{M} = \{X \in \mathrm{Mat}_{2\times 2}(\mathbb{R}) \mid AX = \mathbf{0}\}$ gdje je $A = \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}$. Odredite bazu i dimenziju vektorskog prostora $\mathcal{M}$. Pronađite mu i (neku) bazu za direktni komplement.

2. Kanonska baza za $\mathbb{R}^n$ je skup $\{e_1, e_2, ..., e_n\}$ vektora iz tog prostora takvih da e_i na i-toj poziciji ima broj 1, a na svim ostalim nule. Za linearni operator $T : \mathbb{R}^6 \to \mathbb{R}^4$ zadan matricom u paru kanonskih baza

$$T = \begin{bmatrix} 1 & 1 & 2 & 1 & 2 & 2 \\ 1 & 2 & 2 & 1 & 2 & 2 \\ 2 & 2 & 4 & 2 & 4 & 4 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

odredite jezgru i sliku. Da li je vektor $(0, 0, 0, -1)^\top$ u slici od T?

3. Dat je vektorski potprostor $\mathcal{L}$ vektorskog prostora $\mathrm{Mat}_{2\times 2}(\mathbb{R})$ definisan sa

$$\mathcal{L} = \{A \in \mathrm{Mat}_{2\times 2}(\mathbb{R}) \mid AX - XA = \mathbf{0}, X = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}\}.$$

Razmatrajući standardni unutrašnji proizvod za matrice $\langle A, B \rangle = \mathrm{trag}(A^\top B)$ odrediti ortonormiranu bazu za $\mathcal{L}$.

4. U unitarnom prostoru $\mathcal{P}_2 = \{p(x) = ax^2 + bx + c : a, b, c \in \mathbb{R}\}$ polinoma stepena manjeg ili jednakog 2 sa skalarnim (unutrašnjim) proizvodom $\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)\, \mathrm{d}x$ dat je potprostor

$$\mathcal{M} = \mathrm{span}\{x^2 - 1, x + 1\}.$$

Odredite jednu bazu za $\mathcal{M}^\perp$, te nađite prikaz polinoma $p(x) = 2x^2 + x + 5$ u obliku sume $p = p_1 + p_2$, pri čemu je $p_1 \in \mathcal{M}$, $p_2 \in \mathcal{M}^\perp$.

Važno: Ovaj papir treba predati zajedno s rješenjima zadataka! Svaku formulu koju mislite koristit, u sva 4 zadatka, obavezno napisati, kao i značenja simbola iz formule. Ispit pisati isključivo hemiskom olovkom plave ili crne tinte. Prije rješenja prepisati postavku (tekst) zadatka.

(#) Neka je $\mathcal{M} = \{ X \in Mat_{2\times 2}(\mathbb{R}) \mid AX = \mathbf{0} \}$ gdje je $A = \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}$.
Odredite bazu i dimenziju vektorskog prostora $\mathcal{M}$.
Pronađite mu i (neku) bazu za direktni komplement.

Rj.
Označimo elemente matrice $X \in Mat_{2\times 2}(\mathbb{R})$ sa x_1, x_2, x_3, x_4 tj.

$X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$. Izračunajmo AX

$$AX = \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \begin{pmatrix} 2x_1 + x_3 & 2x_2 + x_4 \\ 6x_1 + 3x_2 & 6x_2 + 3x_4 \end{pmatrix}$$

Primjetimo sad da vektorski prostor $\mathcal{M}$ možemo pisati u drugačijem obliku

$$\mathcal{M} = \left\{ \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in Mat_{2\times 2}(\mathbb{R}) \middle| \; 2x_1 + x_3 = 0,\; 2x_2 + x_4 = 0,\; 6x_1 + 3x_3 = 0,\; 6x_2 + 3x_4 = 0 \right\}$$

Samo za trenutak umjesto prostora $\mathcal{M}$ promatrajmo prostor $\mathcal{M}'$

$$\mathcal{M}' = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 \middle| \; 2x_1 + x_3 = 0,\; 2x_2 + x_4 = 0,\; 6x_1 + 3x_3 = 0,\; 6x_2 + 3x_4 = 0 \right\}$$

$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 \middle| \; \underbrace{\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 6 & 0 & 3 & 0 \\ 0 & 6 & 0 & 3 \end{bmatrix}}_{= A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} = \ker(A)$$

Sad možemo primjetiti da je mnogo lakše odrediti bazu za $\mathcal{M}'$, pa tu dobijenu bazu iskoristiti i pronaći bazu za $\mathcal{M}$.

Pa odredimo bazu za $\ker(A)$.

$$\left[\begin{array}{cccc|c} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 6 & 0 & 3 & 0 & 0 \\ 0 & 6 & 0 & 3 & 0 \end{array}\right] \underset{IV_v + II_v\cdot(-3)}{\overset{III_v + I_v\cdot(-3)}{\sim}} \left[\begin{array}{cccc|c} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \quad \begin{aligned} 2x_1 + x_3 &= 0 \\ 2x_2 + x_4 &= 0 \end{aligned} \qquad \begin{aligned} x_3 &= -2x_1 \\ x_4 &= -2x_2 \end{aligned} \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} t \\ s \\ -2t \\ -2s \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -2 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1 \\ 0 \\ -2 \end{pmatrix} s$$

$s, t \in \mathbb{R}$

$$M' = \operatorname{span}\left(\left\{ \begin{pmatrix} 1 \\ 0 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -2 \end{pmatrix} \right\}\right)$$

Prema tome baza za M je $\left\{ \begin{pmatrix} 1 & 0 \\ -2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix} \right\}$, a $\dim(M) = 2$.

Ostalo je još da odredimo neku bazu za direktni komplement. Nadopunimo bazu od M' do baze za $\mathbb{R}^4$ (prisjetimo se osnovne kolone u A generišu $\operatorname{im}(A)$)

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \dots \sim \begin{bmatrix} 1 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & 0 & -1/2 \\ 0 & 0 & 1 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1/2 \end{bmatrix}$$

Baza za direktni komplement od M je

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}.$$

(#) Kanonska baza za $\mathbb{R}^n$ je skup $\{e_1, e_2, \dots, e_n\}$ vektora iz tog prostora takvih da e_i na i-toj poziciji ima broj 1, a na svim ostalim nule. Za linearni operator $T : \mathbb{R}^6 \to \mathbb{R}^4$ zadan matricom u paru kanonskih baza

$$T = \begin{bmatrix} 1 & 1 & 2 & 1 & 2 & 2 \\ 1 & 2 & 2 & 1 & 2 & 2 \\ 2 & 2 & 4 & 2 & 4 & 4 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

odredite jezgru i sliku. Da li je vektor $(0,0,0,-1)^T$ u slici od T?

R: Prisjetimo se

Ako su $\mathcal{B} = \{u_1, u_2, \dots, u_n\}$ i $\mathcal{B}' = \{v_1, v_2, \dots, v_n\}$ redom baze za $\mathcal{U}$ i $\mathcal{V}$ tada matrica linearne transformacije $T : \mathcal{U} \to \mathcal{V}$ u odnosu na par $(\mathcal{B}, \mathcal{B}')$ je definisana sa

$$[T]_{\mathcal{B}\mathcal{B}'} = \begin{pmatrix} | & | & & | \\ [T(u_1)]_{\mathcal{B}'} & [T(u_2)]_{\mathcal{B}'} & \dots & [T(u_n)]_{\mathcal{B}'} \\ | & | & & | \end{pmatrix}.$$

U našem slučaju ako označimo sa $\mathcal{S} = \left\{ \begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\0\\1\end{pmatrix} \right\}$
i $\mathcal{S}' = \left\{ \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \right\}$ vrijedi

$$[T]_{\mathcal{S}\mathcal{S}'} = \begin{pmatrix} 1 & 1 & 2 & 1 & 2 & 2 \\ 1 & 2 & 2 & 1 & 2 & 2 \\ 2 & 2 & 4 & 2 & 4 & 4 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

Prisjetimo se

Ako je $T \in \mathcal{L}(U,V)$ i ako su $\mathcal{B}$ i $\mathcal{B}'$ redom baze za U i V tada $\forall u \in U$

$$[T(u)]_{\mathcal{B}'} = [T]_{\mathcal{B}\mathcal{B}'}[u]_{\mathcal{B}}$$

U našem slučaju $\forall u \in \mathbb{R}^6$ $[T(u)]_{\varphi'} = T\,[u]_{\varphi}$.

Sad imamo
$$\ker(T) = \{x \in \mathbb{R}^6 \mid T(x) = 0\}$$
$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_6 \end{pmatrix} \in \mathbb{R}^6 \;\middle|\; T\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\}$$

Pa vidimo da je problem traženja jezgra operatora T ekvivalentan problemu traženja jezgra matrice operatora T.

$$\left[\begin{array}{cccccc|c} 1 & 1 & 2 & 1 & 2 & 2 & 0 \\ 1 & 2 & 2 & 1 & 2 & 2 & 0 \\ 2 & 2 & 4 & 2 & 4 & 4 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 \end{array}\right] \sim \dots \sim \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ \left[\begin{array}{cccccc|c} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Prema tome
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} -s \\ 0 \\ -t-u \\ s \\ t \\ u \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} s + \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} u, \quad s,t,u \in \mathbb{R}$$

Prema tome

$$\ker(T) = \operatorname{span}\left\{ \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Slično

$$im(T) = \{T(x) \mid x \in \mathbb{R}^6\} = \{T \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_6 \end{pmatrix} \mid x \in \mathbb{R}^6\}$$

Pa je problem traženja slike operatora T ekvivalentan problemu traženja slike matrice operatora T.

Prisjetimo se

<u>Osnovne kolone u A generišu $im(A)$</u>

Prema tome

$$im(T) = span\left\{\begin{pmatrix}1\\1\\2\\0\end{pmatrix}, \begin{pmatrix}1\\2\\2\\1\end{pmatrix}, \begin{pmatrix}2\\2\\4\\1\end{pmatrix}\right\}$$

Na kraju je ostalo još da provjerimo da li $\begin{pmatrix}0\\0\\0\\-1\end{pmatrix} \in im(T)$, drugim riječima da provjerimo da li postoje realni brojevi α, β i γ t.d.

$$\alpha\begin{pmatrix}1\\1\\2\\0\end{pmatrix} + \beta\begin{pmatrix}1\\2\\2\\1\end{pmatrix} + \gamma\begin{pmatrix}2\\2\\4\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\-1\end{pmatrix}.$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 1 & 2 & 2 & 0 \\ 2 & 2 & 4 & 0 \\ 0 & 1 & 1 & -1 \end{array}\right] \sim \dots \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} \alpha = 2 \\ \beta = 0 \\ \gamma = -1 \end{array}$$

Prema tome vektor $(0,0,0,-1)^T$ pripada slici od T.

(#) Dat je vektorski podprostor $\mathcal{L}$ vektorskog prostora $Mat_{2\times 2}(\mathbb{R})$ definisan sa

$$\mathcal{L} = \left\{ A \in Mat_{2\times 2}(\mathbb{R}) \;\middle|\; AX - XA = 0,\; X = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \right\}.$$

Posmatrajući standardni unutrašnji proizvod za matrice

$$\langle A, B \rangle = trag(A^T B)$$

odrediti ortonormiranu bazu za $\mathcal{L}$.

Rj.

Da bi odredili ortonormiranu bazu za $\mathcal{L}$ prvo je potrebno pronaći bilo kakvu bazu za $\mathcal{L}$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\left.\begin{array}{l} AX = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 2b & a \\ 2d & c \end{bmatrix} \\ XA = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ 2a & 2b \end{bmatrix} \end{array}\right\} \Rightarrow AX - XA = \begin{bmatrix} 2b-c & a-d \\ 2d-2a & c-2b \end{bmatrix}$$

$$AX - XA = 0 \iff \begin{array}{r} 2b - c = 0 \\ a - d = 0 \\ 2d - 2a = 0 \\ c - 2b = 0 \end{array} \iff \underbrace{\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 2 & -1 & 0 \\ 0 & -2 & 1 & 0 \\ -2 & 0 & 0 & 2 \end{bmatrix}}_{=B} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 2 & -1 & 0 \\ 0 & -2 & 1 & 0 \\ -2 & 0 & 0 & 2 \end{bmatrix} \xrightarrow[III_v + II_v]{IV_v + I_v \cdot 2} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{c} a - d = 0 \\ 2b - c = 0 \\ \hline a = d \\ b = \frac{c}{2} \quad c = 2b \end{array}$$

Odavdje sad nije teško vidjeti da se prostor $\mathcal{L}$ može napisati u obliku:

$$\mathcal{L} = \left\{ \begin{bmatrix} d & b \\ 2b & d \end{bmatrix} \middle| \; b, d \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \alpha + \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \beta \;\middle|\; \alpha, \beta \in \mathbb{R} \right\}$$

$$= \operatorname{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \right\}$$

Baza za $\mathcal{L}$ je $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \right\}$.

Sad iskoristimo Gram-Schmidtovu proceduru pa odredimo ortonormiranu bazu za $\mathcal{L}$.

Klasični Gram-Schmidtov algoritam

Za $k=1$: $\quad u_1 \leftarrow \dfrac{x_1}{\|x_1\|}$

Za $k>1$: $\quad u_k \leftarrow x_k - \sum_{i=1}^{k-1} \langle u_i, x_k \rangle u_i$

$$u_k \leftarrow \frac{u_k}{\|u_k\|}$$

$X_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\quad \|X_1\| = \sqrt{\operatorname{trag}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)} = \sqrt{2}$

$$U_1 \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$U_2 \leftarrow X_2 - \langle U_1, X_2 \rangle U_1$$

$$U_2 \leftarrow \frac{U_2}{\|U_2\|}$$

$\langle U_1, X_2 \rangle = \operatorname{trag}(U_1^T X) = \operatorname{trag}([$

$= \operatorname{trag}\left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \right) = 0$

$X_2 = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$

$$U_2 \leftarrow X_2 - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

$$\|U_2\| = \sqrt{\operatorname{trag}\left(\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \right)} = \sqrt{4+1} = \sqrt{5}$$

Ortonormirana baza za $\mathcal{L}$ je

$$\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \frac{1}{\sqrt{5}} \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \right\}.$$

(#) U unitarnom prostoru $\mathcal{P}_2 = \{p(x) = ax^2 + bx + c \mid a,b,c \in \mathbb{R}\}$ polinoma stepena manjeg ili jednakog 2 sa skalarnim (unutrašnjim) proizvodom $\langle p, q \rangle = \int_{-1}^{1} p(x) q(x)\, dx$ dat je podprostor

$$\mathcal{M} = \text{span}\{x^2 - 1, x+1\}$$

Odredite jednu bazu za $\mathcal{M}^\perp$, te nađite prikaz polinoma $p(x) = 2x^2 + x + 5$ u obliku sume $p = p_1 + p_2$, pri čemu je $p_1 \in \mathcal{M}$, $p_2 \in \mathcal{M}^\perp$.

Rj. Prisjetimo se

<u>Ortogonalni komplement</u>
<u>Za podskup $\mathcal{M}$ unitarnog prostora V, ortogonalni komplement $\mathcal{M}^\perp$ od $\mathcal{M}$ je definisan sa</u>

$$\underline{\mathcal{M}^\perp = \{x \in V \mid \langle m, x \rangle = 0 \text{ za } \forall m \in \mathcal{M}\}}$$

Primjetimo da je $\dim(\mathcal{P}_2) = 3$. Kako je

$\dim(\mathcal{M}) = 2$ i $\mathcal{P}_2 = \mathcal{M} \oplus \mathcal{M}^\perp$ to je $\dim(\mathcal{M}^\perp) = 1$.

Da bi odredili $\mathcal{M}^\perp$ dovoljno je pronaći polinom $p(x) = ax^2 + bx + c$ takav da $\langle p, q \rangle = 0 \ \forall q \in \mathcal{M}$.

Zbog $\dim(\mathcal{M}^\perp) = 1$ dobijeni polinom će biti baza za $\mathcal{M}^\perp$.

Da bi odredili polinom $p(x) = ax^2 + bx + c$ za koji vrijedi $\langle p, q \rangle = 0 \ \forall q \in \mathcal{M}$ najjednostavnije je posmatrati bazu za $\mathcal{M}$ tj. skup $\{x^2 - 1, x+1\}$.

$$\langle ax^2+bx+c,\ x^2-1\rangle = 0$$

$$\int_{-1}^{1}(ax^2+bx+c)(x^2-1)\,dx = \int_{-1}^{1}(ax^4+bx^3+\underline{cx^2-ax^2}-bx-c)\,dx = 0$$

$$\int_{-1}^{1}(ax^4+bx^3+(c-a)x^2-bx-c)\,dx = 0$$

$$\frac{a}{5}x^5\Big|_{-1}^{1} + \underbrace{\frac{b}{4}x^4\Big|_{-1}^{1}}_{=0} + (c-a)\cdot\frac{1}{3}x^3\Big|_{-1}^{1} - \underbrace{\frac{b}{2}x^2\Big|_{-1}^{1}}_{=0} - cx\Big|_{-1}^{1} = 0$$

$$\frac{2}{5}a + \frac{2}{3}(c-a) - 2c = 0$$

$$\frac{2}{5}a - \frac{2}{3}a + \frac{2}{3}c - 2c = 0 \Rightarrow \frac{6-10}{15}a + \frac{2-6}{3}c = 0$$

$$-\frac{4}{15}a - \frac{4}{3}c = 0 \quad \ldots(*)$$

$$\langle ax^2+bx+c,\ x+1\rangle = 0$$

$$\int_{-1}^{1}(ax^2+bx+c)(x+1)\,dx = 0$$

$$\int_{-1}^{1}(ax^3+\underline{bx^2}+\underline{\underline{cx}}+\underline{ax^2}+\underline{\underline{bx}}+c)\,dx = 0$$

$$\int_{-1}^{1}(ax^3+(b+a)x^2+(c+b)x+c)\,dx = 0$$

$$\underbrace{\frac{a}{4}x^4\Big|_{-1}^{1}}_{=0} + \frac{b+a}{3}x^3\Big|_{-1}^{1} + \underbrace{\frac{c+b}{2}x^2\Big|_{-1}^{1}}_{=0} + cx\Big|_{-1}^{1} = 0$$

$$\frac{2}{3}b + \frac{2}{3}a + 2c = 0 \Rightarrow \frac{2}{3}a + \frac{2}{3}b + 2c = 0 \quad \ldots(**)$$

Iz (*) i (**) vidimo da imamo sustav od dvije jednačine su tri nepoznate. Jednu nepoznatu uzimamo proizvoljno npr.

$a = 15t, \; t \in \mathbb{R}$

$(*) \Rightarrow \quad -\frac{4}{15} \cdot 15t - \frac{4}{3}c = 0$

$$-4t - \frac{4}{3}c = 0 \Rightarrow \frac{4}{3}c = -4t \Rightarrow c = -3t, \; t \in \mathbb{R}$$

$(**) \Rightarrow \quad \frac{2}{3} \cdot 15t + \frac{2}{3}b + 2 \cdot (-3)t = 0$

$$10t + \frac{2}{3}b - 6t = 0$$

$$\frac{2}{3}b = -4t \quad / \cdot 3 \Rightarrow 2b = -12t \Rightarrow b = -6t$$

Sad $ax^2 + bx + c$ postaje (za $t = 1$) $\quad 15x^2 - 6x - 3 \quad / : 15$

$$x^2 - \frac{6}{15}x - \frac{3}{15}$$

Odavde vidimo da je $\mathcal{M}^\perp = \mathrm{span}\left\{ x^2 - \frac{2}{5}x - \frac{1}{5} \right\}$

baza za $\mathcal{M}^\perp$

Ostalo je još da pronađemo skalare α, β i γ tako da

$$2x^2 + x + 5 = \alpha(x^2 - 1) + \beta(x + 1) + \gamma\left(x^2 - \frac{2}{5}x - \frac{1}{5}\right)$$

$$\begin{aligned} \alpha \qquad + \gamma &= 2 \quad \ldots(1) \\ \beta - \frac{2}{5}\gamma &= 1 \quad \ldots(2) \\ -\alpha + \beta - \frac{1}{5}\gamma &= 5 \quad \ldots(3) \end{aligned}$$

$$\begin{aligned} (1)+(3):& \quad \beta + \frac{4}{5}\gamma = 7 \quad (\mathrm{I}) \\ (2):& \quad \beta - \frac{2}{5}\gamma = 1 \quad (\mathrm{II}) \\ \hline (\mathrm{I})-(\mathrm{II}):& \quad \frac{6}{5}\gamma = 6 \quad / \cdot 5 \\ & \quad 6\gamma = 30 \Rightarrow \gamma = 5 \end{aligned}$$

$$\gamma = 5 \Rightarrow \beta = 7 - 4$$

$$\beta = 3 \Rightarrow \alpha = -3$$

Prema tome

$$2x^2 + x + 5 = \underbrace{(-3)(x^2 - 1) + 3(x + 1)}_{\in \mathcal{M}} + \underbrace{5\left(x^2 - \frac{2}{5}x - \frac{1}{5}\right)}_{\in \mathcal{M}^\perp}$$